Section 11-2, Mathematics 104

Composite and Inverse functions

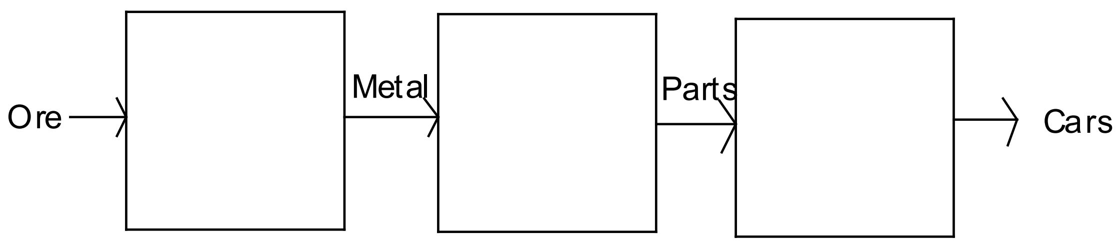
Imagine you have three factories.

The first takes in raw ore and puts out refined metals.

The second takes in refined metals and puts out car parts.

The third takes in the car parts, assembles them and puts out a car.

If you put all three of these factories next to each other, and moved the outputs of the first two factories into the 2nd and third, you would have one single factor that takes in raw ore and outputs cars.



Putting factories together like this is similar to what creating composite functions is like.

If you have two functions:

$$f(x) = x + 5$$
$$g(x) = x^2$$

You can form a composite function where the output of g is the input of f.

This new function can be written as

$$(f \circ g)(x) \text{ or } f(g(x))$$

What this function does is square the input and then add 5 to it.

$$(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 + 5$$

Note that this operation of composition of functions is not commutative.

$$(g \circ f)(x) = g(f(x)) = g(x+5) = (x+5)^2 = x^2 + 10x + 25 \neq x^2 + 5$$

A few examples to try:

$$f(x) = 2x + 4$$
$$g(x) = 3x - 1$$

Find $(f \circ g)(x)$ and $(g \circ f)(x)$

The domain and range of a composition of functions
The domain and range of a composition of functions may not be obvious.

Example:
$$f(x) = x + 1$$
$$g(x) = \sqrt{x}$$

The domain and range of $f(x)$ is all real numbers.
The domain and range of $g(x)$ is all real numbers ≥ 0.

The domain of $(f \circ g)(x) = \sqrt{x} + 1$ is all real numbers ≥ 0. The range is all real numbers ≥ 1

Example:

The domain of $(g \circ f)(x) = \sqrt{x+1}$ is all real numbers ≥ -1.
The range is all real numbers ≥ 0.

One-To-One Functions and their Inverses

What is a one-to-one function

Here is a simple example:

$f\left(x\right)=x+4$

$$\left|\begin{pmatrix}1\\2\\3\end{pmatrix}\right. \to \begin{pmatrix}5\\6\\7\end{pmatrix}$$

What is not a one-to one function?

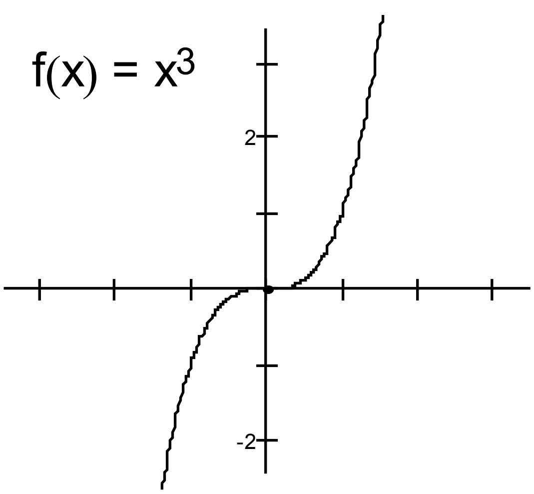
$f\left(x\right)=x^2$

$$\left|\begin{pmatrix}1\\2\\-2\end{pmatrix}\right. \to \begin{pmatrix}1\\4\\4\end{pmatrix}$$

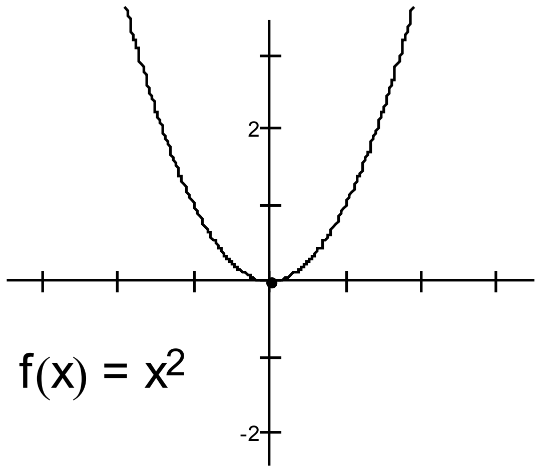
A one-to-one function is one in which each member of the range $f(x)$ has a unique x in the domain.

Another well known one-to-one function is

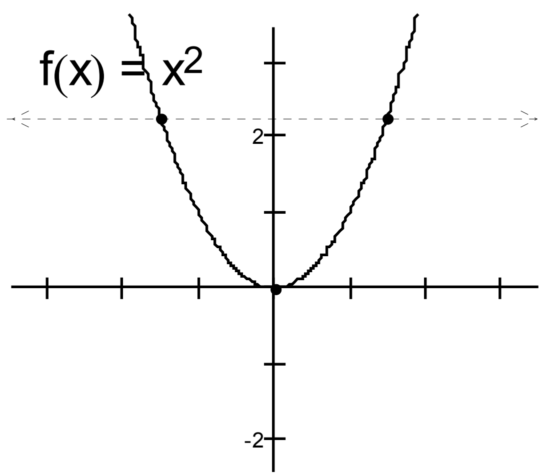
$f\left(x\right)=x^3$



$f(x) = x^2$ is not a one to one function.



Is there an obvious way we can tell from this diagram?



Note that a horizontal line intersecting two points shows that this is NOT a one-to-one function.

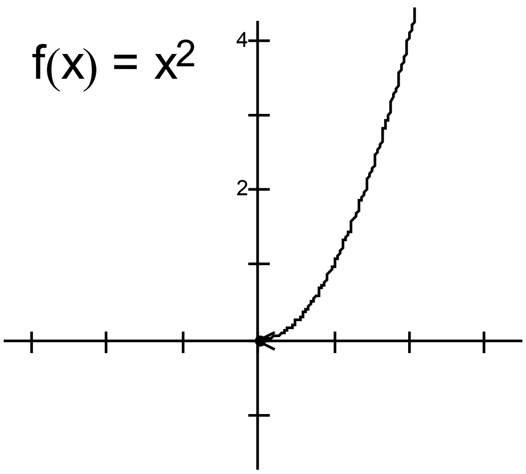
This is called the **Horizontal Line Test**.

It sometimes is possible to make a function that is not one-to-one a one-to-one function by carefully restricting it's Domain.

Here's how to do it with $f(x) = x^2$

Make the $Domain = [0, \infty)$

Now the graph looks like this:



And now it passes the horizontal line test.

Inverse Functions

One very important feature of a one-to-one function is that it has an inverse.
Whenever you have an inverse function, both the function and its inverse are one-to-one.

If we have a function $f(x)$ we write it's inverse as $f^{-1}(x)$.

Do not confuse this with $\left(f(x)\right)^{-1} = \dfrac{1}{f(x)}$

An inverse function takes elements of the range of a function back to the element of the domain that they came from:

$$x \to f(x) \to x$$

$$\left|\begin{pmatrix}1\\2\\3\end{pmatrix} \to \begin{pmatrix}5\\6\\7\end{pmatrix} \to \begin{pmatrix}1\\2\\3\end{pmatrix}\right.$$

An inverse un-does what a function does.

One way to look at this is the composition of a function and its inverse is always *x*.

$$f\left(f^{-1}(x)\right) = x$$

Also

$$f^{-1}\left(f(x)\right) = x$$

For the function $f(x) = x^2$

To find the inverse, we first have to restrict its domain so that the *f* is one-to-one. We do this by setting $Domain = [0, \infty)$

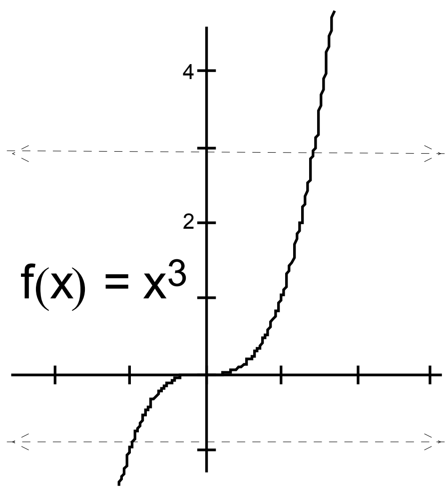
It now has an inverse $f^{-1}(x) = \sqrt{x}$

One way to see if a function is one-to-one is if it passes the horizontal line test

Example:

Note that $f(x) = x^3$ passes the horizontal line test, so its inverse

$f(x) = \sqrt[3]{x}$ has a domain of all real numbers.



How to find an inverse function

There is a simple procedure to find the inverse of a function.

Start with the function.

$f(x) = 2x + 3$

Replace $f(x)$ with y.

$y = 2x + 3$

Switch the x and y, and solve for y.

$$x = 2y + 3$$
$$2y + 3 = x$$
$$2y = x - 3$$
$$y = \frac{x-3}{2}$$

Finally replace y with $f^{-1}(x)$

Example:

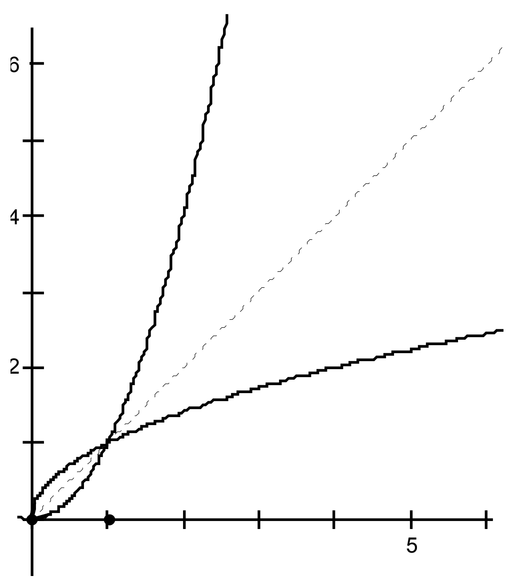
$$f(x) = x^3 + 1$$
$$y = x^3 + 1$$
$$x = y^3 + 1$$
$$y^3 + 1 = x$$
$$y^3 = x - 1$$
$$y = \sqrt[3]{x-1}$$

Graphs of a function and its inverse

There is an interesting graphical relationship between a function and it's inverse.

They are reflections across the line *y*=*x*.

$$f(x) = x^2$$
$$f^{-1}(x) = \sqrt{x}$$



Note that any point on one function, eg. $(2,4)$ means that the point $(4,2)$ will be on inverse function.

Verifying an Inverse Function

One way to verify an inverse function is to check that

$f\left(f^{-1}(x)\right)=x$

Example:

Are

$f(x)=x^2-1$ and $g(x)=\sqrt{x+1}$ inverse functions?

$f(g(x))=f\left(\sqrt{x+1}\right)=\left(\sqrt{x+1}\right)^2-1=x+1-1=x$

Find the inverse of $f(x)=\dfrac{1}{x+2}$